Predefined functions for real and complex numbers in *SMath Studio*
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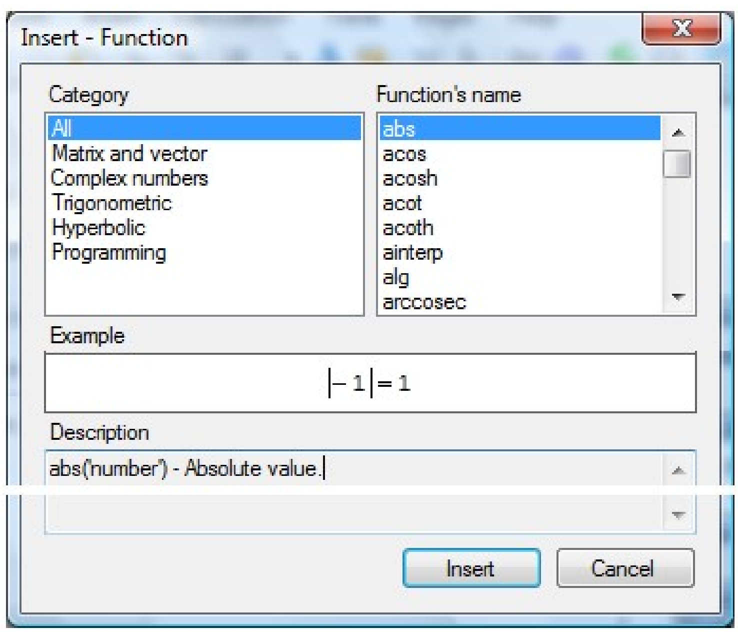
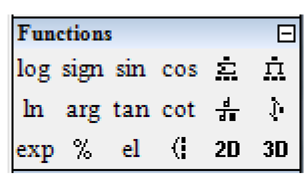
Functions for real numbers - The following functions are available for application to real numbers:

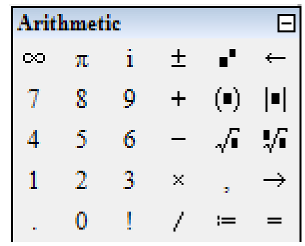
- abs absolute value
- exp exponential function
- Gamma Gamma (Γ) function
- ln natural logarithm, i.e., logarithm of base *e*
- log logarithm of any base
- log10 logarithm of base 10
- mod modulus
- nthroot the *n*-th root of a number
- numden decompose a fraction into numerator and denominator
- perc percentage
- round rounds to an integer
- sign extracts the sign
- sqrt square root
- random generates a random number

These functions are available, unclassified, by using the *Insert* > *Function* menu and then selecting the *All* category of functions:



Some of these functions are also available in the *Functions* palette:
The *Function* palette includes also trigonometric functions (*sin, cos, tan, cot*), calculus expressions (summation, product, derivative, integral), functions that apply to matrices (*el*), and functions that apply to graphs (the last three symbols in the last line).





Some of these functions are also available in the *Arithmetic* palette. These include the absolute value (*abs*), the square root (*sqrt*), and the *n*-th root (*nthroot*) functions. Also shown in the *Arithmetic* palette are the following items:

- Mathematical Constants: Positive infinity (∞), Pi (π), Imaginary unit (i)
- Numerical Digits: 0-9
- Arithmetic operators: ±, +, -, ×, /, power
- Evaluation operators: Definition (:=), Numerical Evaluation (=), Symbolic Evaluation (→)
- Postfix Operators: Factorial (!)
- Editing Characters: Decimal point (.), Comma (,), Backspace (←)

Since trigonometric and hyperbolic functions apply also to real numbers, we provide a list of those functions available under the *Function – Insert* form (see above) under the headings *Trigonometric* and *Hyperbolic:*

Trigonometric:

- sin sine
- cos cosine
- tan tangent
- cot cotangent
- sec secant
- csc cosecant
- asin inverse sine
- acos inverse cosine
- atan inverse tangent
- acot inverse cotanget
- arcsec inverse secant
- arccosec inverse cosecant

Hyperbolic:

- sinh hyperbolic sine
- cosh hyperbolic cosine
- tanh hyperbolic tangent
- coth hyperbolic cotangent
- sech hyperbolic secant
- csch hyperbolic cosecant
- asinh inverse hyperbolic sine
- acosh inverse hyperbolic cosine
- atanh inverse hyperbolic tangent
- acoth inverse hyperbolic cotanget

Examples of functions applied to real numbers

These functions can be inserted from the *Functions – Insert* form (*Insert* > *Function* menu), the *Functions* palette, or simply by typing the name of the function into a region of the *SMath Studio* worksheet. The following are examples of real-number functions in *SMath Studio:*

//EXAMPLES OF FUNCTIONS FOR REAL NUMBERS:		//Using "Ctrl+." instead of "="
//abs: type "abs(-3.25)=" to produce:	$\left\|-3.25\right\|=3.25$	$\left\|-3.25\right\|\rightarrow\frac{13}{4}$
//exp: type "exp(-0.5)=" to produce::	$\exp\left(-0.5\right)=0.6065$	$\exp\left(-0.5\right)\rightarrow\exp\left(-\frac{1}{2}\right)$
//"exp(-0.5)"is same as"e^(-0.5)":	$e^{-0.5}=0.6065$	$e^{-0.5}\rightarrow\frac{1}{\sqrt{e}}$

```
//Gamma: type "Gamma(1.5)=" to produce:      Gamma(1.5)=0.8862
//ln: type "ln(3.2)=" to produce:            ln(3.2)=1.1632
//"exp" and "ln" are inverse functions:      exp(ln(5))=5
//log: type "log(10,2)=" to produce:         log_2(10)=3.3219
//log10: type "log10(8.2)=" to produce:      log10(8.2)=0.9138
//mod: type "mod(18,5)=" to produce:         mod(18, 5)=3
```

$\mathrm{Gamma}(1.5) \rightarrow \frac{886228183571491}{1000000000000000}$

$\ln(3.2) \rightarrow \ln\left(\frac{16}{5}\right)$

$\ln(\exp(-3.2)) = -3.2$

$\log_2(10) \rightarrow \frac{\ln(10)}{\ln(2)}$

$\mathrm{log10}(8.2) \rightarrow \mathrm{log10}\left(\frac{41}{5}\right)$

$\mathrm{mod}(18, 5) \rightarrow 3$

The *mod* function applies to integers only, and it's described in the following example:

```
//Function "mod" calculates the integer residual (r) of the ratio of two integers m,n,
//where m>n and q is the integer quotient, i.e.: m/n = q + r/m. Thus, if m is a multiple
//of n, r = 0, and mod(m,n) = 0. Otherwise, mod(m,n) = r < n. See the following examples:

mod(5, 1)=0    mod(5, 2)=1    mod(5, 3)=2    mod(5, 4)=1    mod(5, 5)=0

// Thus, function "mod" can be used to determine if an integer m is a multiple of
// another integer n, for if that is the case then "mod(m,n) = 0".
```

```
//nthroot: type "nthroot(81,3)=":
```

$\sqrt[3]{81} = 4.3267 \qquad \sqrt[3]{81} \rightarrow \sqrt[3]{81}$

Function *numdem*, shown below, separates a fraction into a numerator and a denominator:

```
//numden: type "numdem(27.4)=":
```

$\mathrm{numden}(27.4) = \begin{pmatrix} 27.4 \\ 1 \end{pmatrix} \qquad \mathrm{numden}(27.4) \rightarrow \begin{pmatrix} 137 \\ 5 \end{pmatrix}$

$$\mathrm{numden}\left(\frac{1+\sqrt{2}}{\sqrt{3}+\sin\left(\frac{\pi}{6}\right)}\right) = \begin{pmatrix} 2.7182 \cdot 10^{15} \\ 1.9604 \cdot 10^{15} \end{pmatrix} \qquad \mathrm{numden}\left(\frac{1+\sqrt{2}}{\sqrt{3}+\sin\left(\frac{\pi}{6}\right)}\right) \rightarrow \begin{pmatrix} 1+\sqrt{2} \\ \sqrt{3}+\sin\left(\frac{\pi}{6}\right) \end{pmatrix}$$

Notice that, in its numerical evaluation, the last example shows both numerator and denominator multiplied by 1015. These two factors obviously cancel when the fraction is put together again, but it serves to emphasize that *SMath Studio* calculates values with 15 decimals.

More functions for real numbers are shown next:

```
//numden: type "numdem(27.4)=":
//perc: type "perc(10,25)=" :
```

$\mathrm{numden}(27.4) = \begin{pmatrix} 27.4 \\ 1 \end{pmatrix} \qquad \mathrm{numden}(27.4) \rightarrow \begin{pmatrix} 137 \\ 5 \end{pmatrix}$

$\mathrm{perc}(10, 25) = 2.5 \qquad \mathrm{perc}(10, 20) \rightarrow \mathrm{perc}(10, 20)$

```
//round: "round(x,n)" rounds up a floating-point value x to n decimal figures:
```

$\text{round}(10.23446, 4) = 10.2345$ $\qquad$ $\text{round}(-3.12567, 4) = -3.1257$

$\text{round}(10.23446, 3) = 10.234$ $\qquad$ $\text{round}(-3.12567, 3) = -3.126$

$\text{round}(10.23446, 2) = 10.23$ $\qquad$ $\text{round}(-3.12567, 2) = -3.13$

$\text{round}(10.23446, 1) = 10.2$ $\qquad$ $\text{round}(-3.12567, 1) = -3.1$

$\text{round}(10.23446, 0) = 10$ $\qquad$ $\text{round}(-3.12567, 0) = -3$

```
// Function "sign(x)" returns the values -1, 0, or 1, depending on whether
// x is negative, zero, or positive, e.g.,
```

$\text{sign}(-3.5) = -1$ $\qquad$ $\text{sign}(0.0) = 0$ $\qquad$ $\text{sign}(3.5) = 1$

```
//sqrt: type "sqrt(23.54)=" to produce:
```

$\sqrt{23.54} = 4.8518$ $\qquad$ $\sqrt{23.54} \rightarrow \frac{\sqrt{1177}}{\sqrt{50}}$

Function *rand* is used to produce random numbers, as indicated below:

```
// Function "rand(x)" produces a random number uniformly distributed between 0 and x.
// The argument "x" must be a positive number. Other examples:
```

$\text{random}(10) = 2$ $\qquad$ $\text{random}(100) = 9$ $\qquad$ $\text{random}(200) = 54$ $\qquad$ $\text{random}(1000) = 951$

```
// Function "random" returns integer numbers. If we were to need a random number
// between 0 and 1, with n decimal figures, use: random(10^n)/10^n, e.g.,
```

$\frac{\text{random}(10^2)}{10^2} = 0.8$ $\qquad$ $\frac{\text{random}(10^3)}{10^3} = 0.321$ $\qquad$ $\frac{\text{random}(10^4)}{10^4} = 0.9049$

```
// To generate a uniformly-distributed random number in the interval [a,b],
// with a<b, use: a + (b-a)*random(10^n)/10^n, where n = 2, 3, 4, ..., e.g.:
```

$$20 + (80 - 20)\cdot\frac{\text{random}(10^3)}{10^3} = 53.6$$

```
// You can write your own function "myrandom" to calculate random numbers:
```

$$\text{myrandom}(a, b, n) := a + \left[(b-a)\cdot\frac{\text{random}(10^n)}{10^n}\right]$$

//Examples: $\text{myrandom}(50, 100, 5) = 76.378$ $\qquad$ $\text{myrandom}(50, 100, 5) = 78.82$

$\text{myrandom}(50, 100, 5) = 84.0755$ $\qquad$ $\text{myrandom}(50, 100, 5) = 69.4585$

```
The following example shows how to produce a row vector of random values in the range
[50,100] using n=5:
```

for $k \in 1..10$

$\quad x_{1\,k} := \text{myrandom}(50, 100, 5)$

```
1) Press "for" in the "Programming" palette
2) Type k in the first place holder in "for"
3) Type "range(1,10)" in the second place holder in "for"
4) Type "x[1,k <space bar> :" below the "for" line
5) Click outside of the region, and type "x="
```

$x = (65.848\ 68.176\ 90.855\ 77.108\ 94.498\ 81.336\ 81.732\ 88.2085\ 90.936\ 66.7025)$

Functions exclusive for complex numbers
The *Function – Insert* form provides the following functions that apply exclusively to complex numbers (let z = *x*+*iy* represent a complex number):

- arg — angle in complex plane, *arg*(*z*) = *atan*(*y*/*x*)
- Im — imaginary part, *Im*(*z*) = *y*
- pol2xy — convert polar coordinates to rectangular coordinates
- Re — real part, *Re*(*z*) = *x*
- xy2pol — convert rectangular coordinates to polar coordinates

The following examples show applications of these functions to complex numbers:

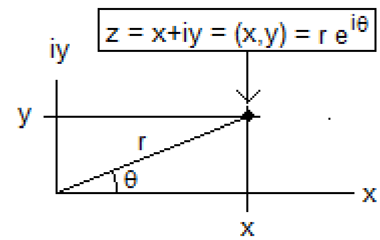
//EXAMPLES OF FUNCTIONS PROPER TO COMPLEX NUMBERS:		//Using "Ctrl+." instead of "="
// arg(z):	$\arg(2+2\cdot i)=0.7854$	$\arg(2+2\cdot i)\rightarrow \arg(2\cdot(1+i))$
//Re(z):	$\mathrm{Re}(-3+5\cdot i)=-3$	$\mathrm{Re}(-3+5\cdot i)\rightarrow \mathrm{Re}(-3+5\cdot i)$
//Im(z):	$\mathrm{Im}(-3+5\cdot i)=5$	$\mathrm{Im}(-3+5\cdot i)\rightarrow \mathrm{Im}(-3+5\cdot i)$
//pol2xy:	$\mathrm{pol2xy}\left(5,\frac{\pi}{6}\right)=\begin{cases}4.3301+1.6653\cdot 10^{-15}\cdot i\\2.5-2.7756\cdot 10^{-15}\cdot i\end{cases}$	$\mathrm{pol2xy}\left(5,\frac{\pi}{6}\right)\rightarrow \mathrm{pol2xy}\left(5,\frac{\pi}{6}\right)$
//xy2pol:	$\mathrm{xy2pol}(3,4)=\begin{cases}5\\0.9273\end{cases}$	$\mathrm{xy2pol}(3,4)\rightarrow \mathrm{xy2pol}(3,4)$

The function *abs,* when applied to a complex number, produces the modulo (length) of the complex number. Function *abs* is not included in the listing of *Complex Numbers* functions in the *Insert – Function* form. However, *abs*, and many other functions that we applied to real numbers above, can be applied to complex numbers as illustrated next:

//EXAMPLES OF FUNCTIONS FOR COMPLEX NUMBERS:		//Using "Ctrl+." instead of "="
//abs(z):	$\lvert 4.5+3.1\cdot i\rvert=5.4644$	$\lvert 4.5+3.1\cdot i\rvert\rightarrow\left\lvert\frac{90+62\cdot i}{20}\right\rvert$
//exp(z):	$\exp(2+3\cdot i)=-7.3151+1.0427\cdot i$	$\exp(2+3\cdot i)\rightarrow \exp(2+3\cdot i)$
//e^z:	$e^{5-3\cdot i}=-146.9279-20.9441\cdot i$	$e^{5-3\cdot i}\rightarrow e^{5-3\cdot i}$
//Gamma(z):	$\mathrm{Gamma}(1.5+2.6\cdot i)=0.0319+0.1071\cdot i$	$\mathrm{Gamma}(1+2\cdot i)\rightarrow\frac{15190422302300700000000}{1000000}$
//ln(z):	$\ln(3+i)=1.1513+0.3218\cdot i$	$\ln(3+i)\rightarrow \ln(3+i)$
//log(z,x):	$\log_2(10+i)=3.3291+0.1438\cdot i$	$\log_2(10)\rightarrow\frac{\ln(10)}{\ln(2)}$
//log10(z):	$\mathrm{log10}(8.2-3\cdot i)=0.9411-0.1523\cdot i$	$\mathrm{log10}(8.2-3\cdot i)\rightarrow \mathrm{log10}\left(\frac{41-15\cdot i}{5}\right)$
//nthroot(z,n):	$\sqrt[3]{4+5\cdot i}=1.7747+0.5464\cdot i$	$\sqrt[3]{4+5\cdot i}\rightarrow\sqrt[3]{4+5\cdot i}$
//perc(p,z):	$\mathrm{perc}(10,25+50\cdot i)=2.5+5\cdot i$	$\mathrm{perc}(10,25+10\cdot i)\rightarrow \mathrm{perc}(10,25+10\cdot i)$
//sqrt(z):	$\sqrt{-5+3\cdot i}=0.6446+2.3271\cdot i$	$\sqrt{-5+3\cdot i}\rightarrow\sqrt{-5+3\cdot i}$

Rectangular and polar representation of complex numbers

A complex number written in the form $z = x + iy$ is in its <u>rectangular</u> (or Cartesian) representation. Thus, it can be written also as the ordered pair (x,y), and be represented in an Argand diagram in which the abscissa is x and the ordinate is iy. An alternative way to represent point (x,y) is through its <u>polar</u> representation whose coordinates are (r,θ). The proper way to write the polar representation of a complex number is through the use of <u>Euler's formula</u>: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. With this result,



$$z = x + iy = r\cos(\theta) + i\,r\sin(\theta) = r(\cos(\theta) + i\sin(\theta)) = r\,e^{i\theta} \quad .$$

SMath Studio provides functions *xy2pol* to convert from rectangular (x,y) into polar (r,θ) coordinates, and *pol2xy* to convert from polar (r,θ) to rectangular (x,y) coordinates. Thus, with these functions one can go easily go from rectangular to polar representations of a complex number, and vice versa.

In the following example we convert from rectangular to polar representations of a complex number:

```
// EXAMPLE: Rectangular to polar

z1:=3+4·i                               // z1 in rectangular form

r1:=|z1|            r1=5                // components of polar form

θ1:=arg(z1)         θ1=0.9273

                  ⎧5
xy2pol(3 , 4)=    ⎨                     // direct way to calculate polar form
                  ⎩0.9273               // with "xy2pol"
```

The following example shows a conversion from polar to rectangular representations of a complex number:

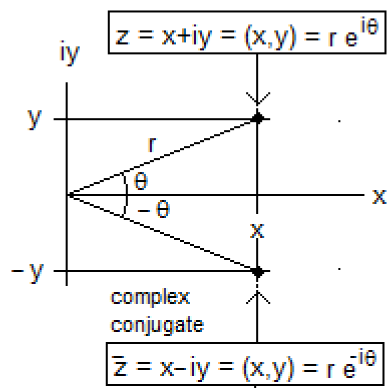
```
// EXAMPLE: Polar to rectangular
          i·π/6
z2:=10·e                          // z2 in polar form

z2=8.6603+5·i                     // shown directly in rectangular form


x2:=Re(z2)     x2=8.6603          // or you can separate components
                                  //  of the rectangular form with Re, Im
y2:=Im(z2)     y2=5


                  ⎧                  -15
                  ⎪8.6603+3.3307·10     ·i     // Alternatively, use
pol2xy(10 , π/6)= ⎨                            // pol2xy to calculate
                  ⎪           -15              // rectangular components
                  ⎩5-5.5511·10    ·i

Note: the imaginary parts of the results from "pol2xy" contain
numbers so small (e.g., 3.3307x10^(-15))that they're basically zero.
```

Operations with complex numbers

The following examples show operations with complex numbers in *SMath Studio*:

```
//Operations with complex numbers:
z1:=3.2-1.5·i              z2:=-5.2+2.2·i
// addition:               z1+z2=-2+0.7·i
// subtraction:            z1-z2=8.4-3.7·i
// multiplication:         z1·z2=-13.34+14.84·i
// division:               z1/z2=-0.6255+0.0238·i
// conjugate:              z1c:=Re(z1)-i·Im(z1)
                           z1c=3.2+1.5·i
```

The complex conjugate of a complex number is the reflection of the number $z = x + iy$ about the x axis, i.e., $\bar{z} = x - iy$. This is illustrated in the figure to the right:




```
//Powers of
//the imaginary
//unit:
(i^2, i^3, i^4, i^5) = (-1, -i, 1, i)
```

All other operations follow the rules of algebra with the caveat that $i^2 = -1$, etc. Other powers of the unit imaginary number are shown in the vector to the left.

Using the conjugate we can write: $z \cdot \bar{z} = r^2$.
This calculation is illustrated below:

```
z0:=-2.5+6.2·i     // number
z0c:=-2.5-6.2·i    // conjugate
z0·z0c=44.69       //number x conjugate
√(z0·z0c)=6.6851
|z0|=6.6851         // abs(z0)
```

The *Gamma* function

Most readers with courses in Algebra and Calculus I will be already familiar with most of the functions for real and complex numbers presented in this document. The *Gamma* function may be an exception, since it is an advanced mathematical function and probably would not have been introduced in those courses. The *Gamma* function is defined by an integral, namely,

$$\Gamma(x)=\int_0^\infty t^{z-1}e^{-t}\,dt \quad .$$

The *Gamma* function is related to the factorial operator as follows: $\Gamma(x+1)=x!$, if *x* is an integer.

The following examples use the *Gamma* function in some calculations:

$$\mathrm{Gamma}(5)=24 \qquad 4!=24$$

$$\mathrm{Gamma}(5.6)=61.554$$

$$\mathrm{Gamma}(3+3\cdot i)=-0.4401-0.0636\cdot i$$

Note: the *Gamma* function currently defined in *SMath Studio 0.85* cannot handle negative arguments, or complex arguments whose real part is negative. For many applications this definition will be fine, but the full definition of the *Gamma* function should be able to handle negative arguments. Based on the paper "*A note on the computation of the convergent Lanczos complex Gamma approximation*" by Paul Godfrey (2001), found in http://home.att.net/~numericana/answer/info/godfrey.htm#lanczoscoeffs , I redefined the *Gamma* function to include negative arguments, as follows:

$$\mathrm{myG}(x):=\mathrm{if}\ x\ge 0 \quad \mathrm{Gamma}(x) \quad \mathrm{else} \quad \frac{\pi\cdot(|x|+1)}{\mathrm{Gamma}(|x|+2)\cdot\sin(\pi\cdot(|x|+1))}$$

The figure to the right also shows some calculations of the modified *Gamma* function, and a graph of the function.

$$\mathrm{myG}(-2.2)=-2.205 \qquad \mathrm{myG}(-2)=1.6983\cdot 10^{14}$$

$$\mathrm{myG}(-1.2)=4.8509 \qquad \mathrm{myG}(-3)=-4.0513\cdot 10^{13}$$

myG(x)

Compare the graph with that shown in the one shown in the wikipedia entry:

http://en.wikipedia.org/wiki/Gamma_function

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