Problem description
Consider the 2nd-order ODE: \(y'' + y' + 3y = \sin(x)\)
subject to the initial conditions: \(y(0) = -1\) \(\text{ and } y'(0) = 1\)

Variable substitution to form a system of ODEs:
This 2nd-order ODE can be converted into a system of two 1st-order ODEs by using the following variable substitution:
\[ u_1 = y, \quad u_2 = y' \]

with initial conditions:
\[ u_1 = -1 \text{ and } u_2 = 1 \text{ at } x = 0. \]

The variable substitution \(u_2 = y'\) is equivalent to:
\[
\frac{du_1}{dx} = u_2
\]
[Eq. 1]

while the ODE is re-written as: \(y'' + y' + 3y = \sin(x)\)

or:
\[
\frac{du_2}{dx} = -u_1' \cdot u_2 - 3u_1 + \sin(x)
\]
[Eq. 2]

The system of equations [Eq. 1] and [Eq. 2] is transformed into the vector ODE:

\[
\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1' \cdot u_2 - 3u_1 + \sin(x) \end{pmatrix}
\]
or,
\[
\frac{du}{dx} = f(x, u), \text{ where } u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and }
\]
\[
f(x, u) = \begin{pmatrix} u_2 \\ -u_1' \cdot u_2 - 3u_1 + \sin(x) \end{pmatrix}
\]

The initial conditions are \(u(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\) at \(x = 0\).

We'll solve the ODEs in the interval: \(0 \leq x \leq 20\) using 100 intervals.
Solution (version 1):

First, define the vector function $f(x, u)$:

$$f(x, u) = \begin{bmatrix} u_2 \\ -u_1 \cdot u_2 - 3 \cdot u_1 + \sin(x) \end{bmatrix}$$

The initial conditions are: $x_s = 0 \quad u_s = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The end of the solution interval is: $x_e = 20$

Use 100 intervals: $n = 100$

Calculate the increment size, $\Delta x$:

$$\Delta x = \frac{x_e - x_s}{n} \quad \Delta x = 0.2$$

Create the $x$ solution vector:

$x_{sol} = \left[ x_s, x_s + \Delta x, \ldots, x_e \right]$

The $y$-solution vector gets initialized as follows:

$u_{sol} = u_s \quad u_{sol} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The following "for" loop calculates the Runge-Kutta algorithm (version 1) to produce the solution:

```plaintext
for k = 1 .. n
    x0 = eval(x_{sol}_{k})
    u0 = eval(col([u_{sol}, k]))
    xM = eval(x0 + \frac{1}{2} \cdot \Delta x)
    K1 = eval(\Delta x \cdot f(x0, u0))
    uM = eval(u0 + \frac{1}{2} \cdot K1)
    K2 = eval(\Delta x \cdot f(xM, uM))
    uM = eval(u0 + \frac{1}{2} \cdot K2)
    K3 = eval(\Delta x \cdot f(xM, uM))
    u1 = eval(u0 + K3)
    x1 = eval(x_{sol}_{k+1})
    K4 = eval(\Delta x \cdot f(x1, u1))
    u1 = eval(u0 + \frac{1}{6} \cdot (K1 + 2 \cdot K2 + 2 \cdot K3 + K4))
    u_{sol} = augment(u_{sol}, u1)
```

After completing the iterative process, the solution is stored in a row vector called "ysol". This vector can be transposed to put together the graph of the two solutions as illustrated here:

$u_{sol} = u_{sol}^T$
The blue line represents $u[1]=y$ while the red line represents $u[2] = dy/dx$.

**Solution (version 2):**

First, define the vector function $f(x,y)$:

$$f(x, u) = \begin{bmatrix} u_2 \\ -u_1 \cdot u_2 - 3 \cdot u_1 + \sin(x) \end{bmatrix}$$

The initial conditions are: $x_s = 0$ $\quad u_s = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The end of the solution interval is: $x_e = 20$

Use 100 intervals: $n = 100$

Calculate the increment size, $\Delta x$:

$$\Delta x = \text{eval}\left(\frac{x_e - x_s}{n}\right) \quad \Delta x = 0.2$$

Create the $x$ solution vector: $x_{sol} = \text{eval}(x_s, x_s + \Delta x..x_e)$

The $y$-solution vector gets initialized as follows:

$$u_{sol} = u_s$$

$u_{sol} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The following "for" loop calculates the Runge-Kutta algorithm (version 1) to produce the solution:
for k∈1..n
    x0 = eval(xsol_k)
    u0 = eval(col[usol, k])
    x1 = eval(x0 + Δx)
    x2 = eval(x0 + 2/3Δx)
    K1 = eval(Δx·f(x0, u0))
    u1 = eval(u0 + 1/3·K1)
    x3 = eval(x1 + 1/3·K1)
    u2 = eval(x1 + 1/3·K2)
    K2 = eval(Δx·f(x13, u13))
    u3 = eval(x13 + 1/3·K2)
    K3 = eval(Δx·f(x23, u23))
    u4 = eval(x23 + 1/3·K3)
    K4 = eval(Δx·f(x1, u1))
    u5 = eval(x1 + 1/3·K4)
    usol = augment(usol, u1)

After completing the iterative process, the solution is stored in a row vector called "ysol". This vector can be transposed to put together the graph of the two solutions as illustrated here:

\[
\begin{bmatrix}
\text{usol} & \text{usol}^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{N1} & \text{augment(xsol, col[usol, 1])}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{N2} & \text{augment(xsol, col[usol, 2])}
\end{bmatrix}
\]

The blue line represents \( u[1] = y \) while the red line represents \( u[2] = dy/dx \).